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GAP IN A CONTINUOUS SPECTRUM OF AN ELASTIC WAVEGUIDE WITH A PARTLY CLAMPED SURFACE

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A periodic elastic waveguide whose continuous spectrum contains a gap (interval that can contain a discrete spectrum only) is constructed. The gap prevents wave propagation in the corresponding range of frequencies, which can be used in designing filters and dampers of elastic waves.

Key words: *elastic waveguide, gap, continuous spectrum.*

1. Formulation of the Problem. Let Π be a periodic elastic waveguide made of an anisotropic inhomogeneous material with a bounded periodicity cell ϖ located in the layer $\{x = (x_1, x')\colon\ x_1 \in (0, l),\ x' = (x_2, x_3) \in \mathbb{R}^2\}$ (Fig. 1). By means of scaling, we reduce the cell size l to a unit size and introduce its integer shifts

$$\varpi_j = \{x\colon (x_1 - j, x') \in \varpi\}. \tag{1.1}$$

The interior of the union of cell closures

$$\overline{\Pi} = \bigcup_{j=-\infty}^{+\infty} \overline{\varpi_j} \tag{1.2}$$

is assumed to be a domain with the Lipschitz boundary $\partial\Pi$, in particular, a connected set. We represent the side surface $\gamma^0 = \{x \in \partial\varpi\colon\ x_1 \in (0, 1)\}$ of the cell as the union of the surfaces γ^σ and γ^u, and then introduce periodic sets Γ^σ and Γ^u, using equalities similar to Eqs. (1.1) and (1.2). Let us assume that the boundary Γ^σ is free from external loads, and the boundary Γ^u is rigidly clamped. In this study, we consider the case where both γ^u and γ^σ have a positive area. The cell shape and the representation $\gamma^\sigma \cup \gamma^u$ of its side surface will be defined in Sec. 5.

We use the matrix form of writing the constitutive relations of three-dimensional anisotropic elasticity theory [1, 2], namely, in a fixed Cartesian coordinate system x, we interpret the vector of displacements $u(x)$ as a column $(u_1(x), u_2(x), u_3(x))^{\rm t}$ (“t” is the sign of transposition). The column of strains

$$\varepsilon = (\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \sqrt{2}\,\varepsilon_{23}, \sqrt{2}\,\varepsilon_{31}, \sqrt{2}\,\varepsilon_{12})^{\rm t} \tag{1.3}$$

is found by the formula $\varepsilon(u; x) = D(\nabla_x)u(x)$, where

$$\nabla_x = (\partial_1, \partial_2, \partial_3)^{\rm t}, \qquad \partial_j = \frac{\partial}{\partial x_j}, \qquad j = 1, 2, 3,$$

$$D(\nabla_x)^{\rm t} = \begin{pmatrix} \partial_1 & 0 & 0 & 0 & 2^{-1/2}\,\partial_3 & -2^{-1/2}\,\partial_2 \\ 0 & \partial_2 & 0 & -2^{-1/2}\,\partial_3 & 0 & 2^{-1/2}\,\partial_1 \\ 0 & 0 & \partial_3 & 2^{-1/2}\,\partial_2 & -2^{-1/2}\,\partial_1 & 0 \end{pmatrix}. \tag{1.4}$$

Note that the factors $\sqrt{2}$ are introduced into Eqs. (1.3) and (1.4) to equalize the natural norms of the tensors and the corresponding columns. After substitution of these factors [2], the orthogonal transformations of the coordinate system give rise to the orthogonal transformations of the column of strains $\varepsilon(u)$ and a similar column of stresses $\sigma(u)$ with six components, which is determined by Hooke’s law

$$\sigma(u; x) = A(x)\varepsilon(u; x).$$

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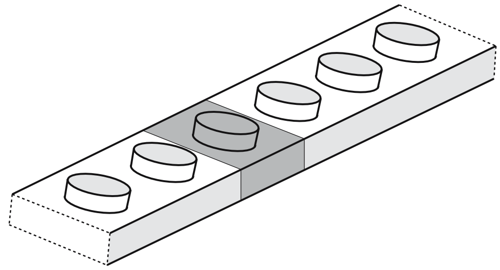

Fig. 1. Periodic waveguide (the shaded area is a cell of the periodic waveguide).

Here, A is a symmetric 6×6 matrix of elastic moduli, which is measurable and satisfies the conditions of positiveness and boundedness for almost all $x \in \Pi$:

$$c_A|a|^2 \le a^{\mathrm{t}} A(x) a \le C_A|a|^2, \qquad a \in \mathbb{R}^6, \qquad C_A \ge c_A > 0. \tag{1.5}$$

The elastic waves in a periodic cylinder Π are described by the following boundary-value problem:

$$\begin{gathered} D(-\nabla_x)^{\mathrm{t}} A(x) D(\nabla_x) u(x) = \lambda \rho(x) u(x), \qquad x \in \Pi, \\ D(n(x))^{\mathrm{t}} A(x) D(\nabla_x) u(x) = 0, \quad x \in \Gamma^\sigma, \qquad u(x) = 0, \quad x \in \Gamma^u. \end{gathered} \tag{1.6}$$

Here, $n = (n_1, n_2, n_3)^{\mathrm{t}}$ is the unit vector (column) of the external normal (by virtue of the Lipschitz surface, it is determined for almost all $x \in \partial\Pi$), λ is the spectral parameter (squared frequency), and ρ is the material density, i.e., a measurable function satisfying the condition of positiveness and boundedness

$$0 < c_\rho \le \rho(x) \le C_\rho \tag{1.7}$$

for almost all $x \in \Pi$. At the infinity, the physical characteristics of the waveguide only insignificantly differ from periodic, namely, the following inequality is valid:

$$|A(x) - A^0(x)| + |\rho(x) - \rho_0(x)| \le c \exp(-\delta_0|z|), \qquad \delta_0 > 0$$

(A^0 and ρ_0 are the matrix and the scalar 1-periodically depending on the longitudinal variable x_1 and inheriting all properties of A and ρ).

The results of this work are valid for all perturbations of the solid in a bounded area. To simplify the considerations, however, we pose problem (1.6) for a periodic solid (1.2). Using the changes $A \mapsto A^0$ and $\rho \mapsto \rho_0$, we obtain the following periodic problem:

$$\begin{gathered} D(-\nabla_x)^{\mathrm{t}} A^0(x) D(\nabla_x) u(x) = \lambda \rho^0(x) u(x), \qquad x \in \Pi, \\ D(n(x))^{\mathrm{t}} A^0(x) D(\nabla_x) u(x) = 0, \quad x \in \Gamma^\sigma, \qquad u(x) = 0, \quad x \in \Gamma^u. \end{gathered} \tag{1.8}$$

The essential spectra of problems (1.6) and (1.8) coincide (see Sec. 4).

2. Continuous Spectrum and Elastic Waves. If Π is a cylinder $\omega \times \mathbb{R}$ (ω is a bounded domain in the plane $\mathbb{R}^2$), and the stiffness matrix A^0 and density ρ_0 are independent of the longitudinal variable z, there exists the only threshold $\lambda^\dagger$ at which the half-interval $[0, \lambda^\dagger)$ contains only the discrete spectrum of problem (1.6), and the ray $[\lambda^\dagger, +\infty)$ can contain only the continuous spectrum. The eigenvalues λ_n of problem (1.6) correspond to eigenvector functions possessing a finite elastic energy (the so-called trapping modes) [3, 4], and the points λ_n can be located either below or above the threshold $\lambda^\dagger$ (see, e.g., [4–13]). If $\lambda \in [0, \lambda^\dagger)$ is not an eigenvalue, then the inhomogeneous problem (1.6) is uniquely solvable in the natural energy class $\overset{0}{H}{}^1(\Pi; \Gamma^u)$ [vector functions from the Sobolev space $H^1(\Pi)$ vanishing on Γ^u]. At the same time, in the case $\lambda \ge \lambda^\dagger$, the problem in the class $\overset{0}{H}{}^1(\Pi; \Gamma^u)$ is ill-posed and requires certain radiation conditions to be imposed at infinity, which would allow distinguishing of incoming and outgoing waves. In other words, wave processes arise on the continuous spectrum, i.e., above the threshold, while there are no wave processes on the resolvent field or discrete spectrum, i.e., below the threshold.

The Floquet–Bloch theory (see, e.g., [14–16]) predicts that specific gaps may appear on the continuous spectrum for periodic waveguides. These gaps are intervals whose ends are points of the continuous spectrum, but these gaps can contain only the discrete spectrum of problem (1.6). Thus, in contrast to the cylinder $\omega \times \mathbb{R}$, the continuous spectrum of the periodic elastic solid (1.2) is not necessarily a connected set, and wave propagation in some frequency ranges is impossible even above the first threshold. This property of periodic waveguides can be used in the development of filters and dampers of elastic waves.

The absence of gaps is not a specific feature of cylindrical solids: a periodic solid can also have a continuous spectrum, for instance, a small perturbation of the boundary of the cylindrical elastic cell $\omega \times (0, 1)$ is not able to open a gap. The author is not aware of any papers where the gap is found on a continuous spectrum of a periodic elastic waveguide. Examples of particular solids with this property is the basic result of this work, which was announced in [17]. It should be noted that previous publications on gap opening were mainly based on considering scalar equations in an unbounded space (see [18–24]) or a system of Maxwell equations (see [25]).

3. Spectral Problem on a Cell. Assuming that all data of the problem are rather smooth, we substitute the expression for the Floquet wave into the equations and the boundary conditions (1.8):

$$u(x_1, x') = \exp(i\eta x_1) U(x_1, x'). \tag{3.1}$$

Here, i is the imaginary unity, η is a numerical parameter, and U is the vector function 1-periodic in the variable x_1. As a result, we obtain the following problem on the periodicity cell:

$$\begin{gathered} D(-\partial_1 - i\eta, -\partial_2, -\partial_3)^{\mathrm{t}} A^0(x_1, x') D(\partial_1 + i\eta, \partial_2, \partial_3) U(x_1, x') = \lambda \rho_0(x_1, x') U(x_1, x'), \\ (x_1, x') \in \varpi, \\ D(n(x_1, x'))^{\mathrm{t}} A^0(x_1, x') D(\partial_1 + i\eta, \partial_2, \partial_3) U(x_1, x') = 0, \qquad (x_1, x') \in \gamma^\sigma, \\ U(x_1, x') = 0, \qquad (x_1, x') \in \gamma^u, \\ U(0, x') = U(1, x'), \qquad \partial_1 U(0, x') = \partial_1 U(1, x'). \end{gathered} \tag{3.2}$$

As the system of differential equations contains the square of the spectral parameter η, the operator of problem (3.2) is called the quadratic pencil. The theory of holomorphic pencils was given in [26]. In particular, it follows from the general results that, for all values of λ in the complex plane $\mathbb{C}$, problem (3.2) has a countable set $\{\eta_p\}$ of normal eigenvalues with the only accumulation point at infinity. In addition to the pair $\{\eta, U\}$, the pairs $\{\eta \pm 2\pi, \exp(\mp 2\pi i x_1) U\}$ also satisfy problem (3.2); in other words, the spectrum of problem (3.2) is invariant to shifts along the real axis to distances $\pm 2\pi$, i.e., it is a periodic set (see more details in [27; 28, § 3.4]).

It was found [27] (see also [28]) that, for a certain fixed value of λ, the operator of the inhomogeneous problem (1.6) is the Fredholm operator (has a finite-dimensional kernel and co-kernel, and also a closed image) if and only if the half-segment $[0, 2\pi)$ and, as a consequence (by virtue of periodicity), the entire real axis $\mathbb{R} \subset \mathbb{C}$ are free from eigenvalues η_p of pencil (3.2). The Fredholm nature of the operator means that the point λ belongs either to the discrete spectrum or to the resolvent set [on the latter, problem (1.6) is uniquely solvable].

To use the results obtained, let us give the exact meaning to the objects introduced.

4. Operator Formulation of the Problem. Because of possible irregularities of the boundary and coefficients, the solution of the boundary-value problem (1.2) may fail to be smooth; therefore, the problem should be reformulated as an integral identity [29]:

$$(AD(\nabla_x)u, D(\nabla_x)v)_\Pi = \lambda(\rho u, v)_\Pi, \qquad v \in \overset{0}{H}{}^1(\Pi; \Gamma^u). \tag{4.1}$$

Here, $(\,\cdot\,,\,\cdot\,)_\Pi$ is the scalar product in the Lebesgue space $L_2(\Pi)$. Let us supply the Sobolev space $\mathcal{H} = \overset{0}{H}{}^1(\Pi; \Gamma^u)$ with a specific scalar product

$$\langle u, v \rangle = (AD(\nabla_x)u, D(\nabla_x)v)_\Pi. \tag{4.2}$$

In the Hilbert space $\mathcal{H}$, we introduce the operator $\mathcal{K}$:

$$\langle \mathcal{K}u, v\rangle = (\rho u, v)_\Omega, \qquad u \in \mathcal{H}, \quad v \in \mathcal{H}. \tag{4.3}$$

Replacement of the spectral parameter

$$\mu = \lambda^{-1} \tag{4.4}$$

and simple algebraic transformations allow us to turn the integral identity (4.1) into an abstract spectral equation

$$\mathcal{K}u = \mu u \in \mathcal{H}. \tag{4.5}$$

It is clear that the operator $\mathcal{K}$ is continuous and symmetric; hence, it is self-adjoint. This operator is positive; let us denote its norm by k. The spectrum of the operator $\mathcal{K}$ is located on the segment $[0, k]$ of the real axis $\mathbb{R} \subset \mathbb{C}$. As the domain Π is unbounded, the operator is not compact and, therefore, in addition to the point $\mu = 0$, has a continuous spectrum on the half-interval $(0, k]$ (see [30, § 9.2]).

By virtue of relation (4.4), the λ-spectrum of the variational problem (4.1) [or the boundary-value problem (1.6) for smooth data] inherits all properties of the μ-spectrum of Eq. (4.5), except for those inherent at the point $\mu = 0$ correlated with an infinitely distant point λ.

We can easily show that problem (3.2) with a fixed real η is formally self-adjoint. In the space $H(\eta) = \overset{0}{H}{}^1_{\mathrm{per}}(\varpi; \gamma^u)$ of complex-valued vector functions $U \in H^1(\varpi)$ that vanish on the surface γ^u and are 1-periodic in the variable x_1, by analogy with Eqs. (4.2) and (4.3), we introduce the scalar product

$$\langle U, V\rangle_\eta = (A^0 D(\partial_1 + i\eta, \partial_2, \partial_3)U, D(\partial_1 + i\eta, \partial_2, \partial_3)V)_\varpi \tag{4.6}$$

and the self-adjoint compact positive operator $K(\eta)$:

$$\langle K(\eta)U, V\rangle_\eta = (\rho^0 U, V), \qquad U \in H(\eta), \quad V \in H(\eta). \tag{4.7}$$

We denote the norm of this operator by $k(\eta)$. According to [30] (see Theorems 9.2.1 and 10.2.2), the spectrum of the operator $K(\eta)$ consists of the point $M_\infty = 0$, which belongs to the essential spectrum, and an infinitely small sequence of eigenvalues

$$k(\eta) = M_1(\eta) \geq M_2(\eta) \geq \ldots \geq M_n(\eta) \ldots \to +0 \tag{4.8}$$

that form the discrete spectrum of the operator. The eigenvalues are indicated in formula (4.8) with allowance for their multiplicities; they continuously and 2π-periodically depend on the parameter $\eta \in \mathbb{R}$.

The eigenvalues of problem (4.1)

$$\Lambda_n(\eta) = M_n(\eta)^{-1}$$

[see Eq. (4.4)] form an infinitely large sequence

$$0 < \Lambda_1(\eta) \leq \Lambda_2(\eta) \leq \ldots \leq \Lambda_n(\eta) \leq \ldots \to +\infty \tag{4.9}$$

and also continuously and 2π-periodically depend on η. Therefore, each of the quantities $\Lambda_n(\eta)$ fills a closed segment

$$t_n = \{\Lambda > 0\colon \Lambda = \Lambda_n(\eta),\ \eta \in [0, 2\pi)\}. \tag{4.10}$$

For some values of n, segment (4.10) can degenerate into a point, but examples of such a situation are not known in the elasticity theory. We denote the union of segments (4.10) by T. According to the results of [27] mentioned in Sec. **3**, the set T contains all points for which the operator of the inhomogeneous problem (4.1) [for smooth data, also the operator of problem (1.6)] is no longer the Fredholm operator, namely, the homogeneous model problem acquires a non-decaying solution: Floquet wave (3.1). Correspondingly, the set $S \cup \{M_\infty\}$, where

$$S = \{\mu > 0\colon \mu^{-1} \in T\}, \tag{4.11}$$

is the essential spectrum of the operator $\mathcal{K}$.

The structure of set (4.11) does not eliminate the emergence of gaps (see Fig. 2, which shows the segments $t_1, t_2, \ldots, t_5$, and t_6, and also the gap l_{56}).

In accordance with the standard scheme (see, e.g., [16, 28] and other publications), the operator $\tilde{\mathcal{K}}$ of problem (1.6) can be expressed via the operator $\mathcal{K}$ and the sum of the arbitrarily small and compact operators; therefore, $\tilde{\mathcal{K}}$ and $\mathcal{K}$ have identical essential spectra.

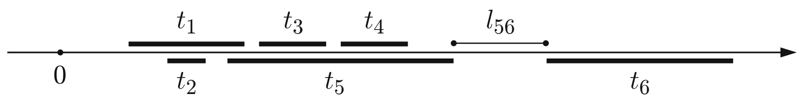

Fig. 2. Gap between the fifth and sixth segments of the continuous spectrum.

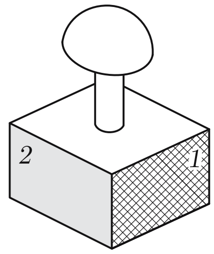

Fig. 3. Mushroom-shaped periodicity cell.

5. Korn Inequality. To verify the existence of a gap in the continuous spectrum of the waveguide with the periodicity cell shown in Fig. 3, we have to use a special (weighted and anisotropic) Korn inequality, which allows us to estimate the bounds of the interval where eigenvalues (4.9) vary. The scheme used to prove this inequality admits generalizations (see below), but we consider symmetric shapes and the clamped surface in the form of a band to simplify the description.

Let $\Xi \subset \mathbb{R}^3$ be a domain with the Lipschitz boundary $\partial\Xi$ and compact closure $\overline{\Xi} = \Xi \cup \partial\Xi$, and Γ be a part of the surface $\partial\Xi$ that has a positive area. The proof of the Korn inequality

$$\|\nabla_x u; L_2(\Xi)\|^2 + \|u; L_2(\Xi)\|^2 =: \|u; H^1(\Xi)\|^2 \leq C(\Xi, \Gamma)\|D(\nabla_x)u; L_2(\Xi)\|^2,$$
$$u \in \overset{0}{H}{}^1(\Xi; \Upsilon) \tag{5.1}$$

was given in [2, 31–33] and other publications. The dependence of the Korn constant C on the shape of the body Ξ and rigidly clamped surface Γ is rather complicated and is not known in the general case.

Let us note the following two facts. First, substitution of the vector function (3.1) with the term $U \in \overset{0}{H}{}^1(\varpi; \gamma^u)$ into inequality (5.1) with $\Xi = \varpi$ and $\Gamma = \gamma^u$ yields the relation

$$(4\pi)^{-2}\,\|U; H^1(\varpi)\|^2 \leq \|u; H^1(\varpi)\|^2 \leq c(\varpi, \gamma^u)\|D(\nabla_y, \partial_z + i\eta)U; L_2(\varpi)\|$$
$$\leq C(\varpi, \gamma^u)\, c_{A^0}^{-1}(A^0 D(\nabla_y, \partial_z + i\eta)U, D(\nabla_y, \partial_z + i\eta)U)_\varpi. \tag{5.2}$$

Second, by virtue of the positiveness of condition (1.5), summation of inequalities (5.1) where $\Xi = \varpi_j$, $\Gamma = \gamma_j^u$, and the constant $C(\varpi_j, \gamma_j^u) = C(\varpi, \gamma^u)$ is independent of the number j of cell (1.1) yields the estimate

$$\|u; H^1(\Pi)\|^2 \leq C(\varpi, \gamma^u)\|D(\nabla_x)u; L_2(\Pi)\|^2 \leq C(\varpi, \gamma^u)c_A^{-1}(AD(\nabla_x)u, D(\nabla_x)u)_\Pi. \tag{5.3}$$

Let the mushroom-shaped periodicity cell shown in Fig. 3

$$\varpi = \Xi \cup \Upsilon_r \cup \Omega \tag{5.4}$$

consist of a massive "mycelium" Ξ in the form of a parallelepiped, $\Xi = \theta \times (-2H, 0)$, and a massive "cap" Ω in the form of a half-ball $\{x,\ (x_1 - 1/2)^2 + x_2^2 + (x_3 - R)^2 < R^2/4,\ x_3 > R\}$, which are connected by a thin "stem" $\Upsilon_r = \omega_r \times (-H, 5R/4)$ "ingrown" both into Ξ and into Ω. Here, $\theta = (-1/2, 1/2) \times (0, 1)$ is a square, $\omega_r = \{(x_1, x_2)\colon (x_1 - 1/2)^2 + x_2^2 < r^2\}$ is a circle of a small radius $r \in (0, R)$, $H > 0$ and $R \in (0, 1/2)$ are fixed sizes. The sets Ξ and Ω, and also the cross section ω_r of the cylinder Υ_r can be rather arbitrary, and their particular shapes are chosen here only to establish the correspondence with Fig. 3. It is only important that massive solids are connected by a thin "ligament" and $\gamma^u = \Gamma = (0, 1) \times \{1/2\} \times (-2H, 0)$, i.e., some part of the surface of one of the bodies (face 1 of the parallelepiped in Fig. 3) is rigidly clamped and, hence, the Korn inequality (5.1) is valid. Note that periodicity conditions are set on face 2 of the parallelepiped and on the opposite face (see Fig. 3).

Let $u \in \overset{0}{H}{}^1(\varpi; \gamma^u)$, $w = \chi_r u$, and $\chi_r \in C^\infty(\mathbb{R})$ be a cut-off function equal to unity at $x_3 > r - H$ and to zero at $x_3 \leq -H$; thereby, $0 \leq \chi_r(x_3) \leq 1$ and $|\partial_3 \chi_r(x_3)| \leq cr^{-1}$. Then, we have

$$\|D(\nabla_x)w; L_2(\Upsilon_r)\|^2 \leq 2\|D(\nabla_x)u; L_2(\Upsilon_r)\|^2 + cr^{-2}\|u; L_2(\omega_r \times (-H, r-H))\|^2. \tag{5.5}$$

This relation together with the Korn inequality (5.1) and the corollary of the one-dimensional Hardy inequality

$$r^{-2}\|u; L_2(\omega_r \times (-H, r-H)\|^2 \leq c\|\varrho_H^{-1}u; L_2(\omega_h \times (-H, r-H)\|^2 \leq c\|\varrho_H^{-1}u; L_2(\Xi)\|^2 \leq c\|u; H^1(\Xi)\|^2,$$

$$\varrho_H(x) = ((x_1 - 1/2)^2 + x_2^2 + (x_3 - H)^2)^{1/2}$$

establishes the inequality

$$\|D(\nabla_x)w; L_2(\Upsilon_r)\|^2 \leq c\|D(\nabla_x)u; L_2(\Upsilon \cup \Xi)\|^2. \tag{5.6}$$

As $w(x) = 0$ at $(x_1, x_2) \in \omega_r$ and $x_3 = -H$, we use the anisotropic Korn inequality for a rod (see [34, 35] and also [36, § 2] and [2, chapter 3]), which was derived in an asymptotically exact manner by means of distributing the weight factors between the components in the Sobolev norm:

$$cr^2\|w; H^1(\Upsilon_r)\|^2 \leq \sum_{p=1}^{2} \int\limits_{\Upsilon_r} \Big(\Big|\frac{\partial w_p}{\partial x_p}\Big|^2 + \Big|\frac{\partial w_3}{\partial x_3}\Big|^2 + r^2\Big(\Big|\frac{\partial w_p}{\partial x_3}\Big|^2 + \Big|\frac{\partial w_3}{\partial x_p}\Big|^2\Big)$$

$$+ r^2\Big(\Big|\frac{\partial w_1}{\partial x_2}\Big|^2 + \Big|\frac{\partial w_2}{\partial x_1}\Big|^2\Big) + r^2|w_p|^2 + |w_3|^2\Big)\, dx \leq c\|D(\nabla_x)w; L_2(\Upsilon_r)\|^2. \tag{5.7}$$

It should be noted that the constants in all estimates are independent of the small parameter $r \in (0, R]$ and trial functions.

Let us consider the field of displacements in the "cap" Ω and present it in the form

$$u(x) = u^\perp(x) + d(x_1 - 1/2, x_2, x_3 - R)b, \tag{5.8}$$

where $b \in \mathbb{R}^6$ and $d(x_1 - 1/2, x_2, x_3 - R)b$ is the rigid motion, i.e., $d(x)$ is a matrix of dimension 3×6 similar to Eq. (1.4):

$$d(x) = \begin{pmatrix} 1 & 0 & 0 & 0 & 2^{-1/2}x_3 & -2^{-1/2}x_2 \\ 0 & 1 & 0 & -2^{-1/2}x_3 & 0 & 2^{-1/2}x_1 \\ 0 & 0 & 1 & 2^{-1/2}x_2 & -2^{-1/2}x_1 & 0 \end{pmatrix}. \tag{5.9}$$

Note that the point $(1/2, 0, R)$ coincides with the center of the half-ball Ω. The term $u^\perp$ satisfies six orthogonality conditions

$$\int\limits_{\Omega} d\Big(x_1 - \frac{1}{2}, x_2, x_3 - R\Big)^{\mathrm{t}} u^\perp(x)\, dx = 0 \in \mathbb{R}^6 \tag{5.10}$$

that eliminate the rigid displacement; therefore, the following variant of the Korn inequality is valid (see, e.g., [33] and also [2, § 2.2]):

$$\|u^\perp; H^1(\Omega)\|^2 \leq c_\Omega\|D(\nabla_x)u^\perp; L_2(\Omega)\|^2 = c_\Omega\|D(\nabla_x)u; L_2(\Omega)\|^2. \tag{5.11}$$

The last equality is satisfied because $D(\nabla_x)\, d(x)$ is the zero matrix of dimension 6×6 in accordance with definitions (1.4) and (5.9) [the column of strains (1.4) degenerates on rigid displacements].

Now we have to estimate the column $b \in \mathbb{R}^6$ in expansion (5.8), which is the solution of the algebraic system

$$\mathcal{M}b = \int\limits_{\Upsilon_r^0} d\Big(x_1 - \frac{1}{2}, x_2, x_3 - R\Big)^{\mathrm{t}} \Big(w(x) - u^\perp(x)\Big)\, dx \in \mathbb{R}^6 \tag{5.12}$$

with a symmetric Gram matrix

$$\mathcal{M} = \int\limits_{\Upsilon_r^0} d\Big(x_1 - \frac{1}{2}, x_2, x_3 - R\Big)^{\mathrm{t}} d\Big(x_1 - \frac{1}{2}, x_2, x_3 - R\Big)\, dx \tag{5.13}$$

constructed with the help of a scalar product in $L_2(\Upsilon_r^0)$ from the columns of the matrix $d(x_1-1/2, x_2, x_3-R)$. As the columns are linearly independent, the matrix $\mathcal{M}$ is positively determined. We use a straight circular cylinder $\Upsilon_r^0=\{x\in\Upsilon_r\colon\ x_3\in(R,5R/4)\}$ as the domain of integration in Eq. (5.13). As the body Υ_r^0 is symmetric, the Gram matrix is a diagonal matrix:

$$\mathcal{M}=\operatorname{diag}\Big\{\frac{\pi}{4}Rr^2, \frac{\pi}{4}Rr^2, \frac{\pi}{4}Rr^2, \frac{\pi}{32}Rr^2\Big(\frac{R^2}{12}+r^2\Big), \frac{\pi}{32}Rr^2\Big(\frac{R^2}{12}+r^2\Big), \frac{\pi}{16}Rr^4\Big\}. \tag{5.14}$$

The first three elements in the right side of Eq. (5.14) are the volume of the body Υ_r^0, and the remaining three elements are the moments of inertia of the body.

By virtue of inequalities (5.5)–(5.7) and (5.11), and also relations (5.12) and (5.14) at $p=1,2$, we find

$$|b_p|^2\le cr^{-4}\Big(\int\limits_{\Upsilon_r^0}(|w_p(x)|^2+|u_p^\perp(x)|^2)\,dx\Big)^2\le cr^{-4}\mathrm{mes}_3(\Upsilon_r^0)\Big(\|w_p;L_2(\Upsilon_r^0)\|^2+\|u_p^\perp;L_2(\Upsilon_r^0)\|^2\Big)$$

$$\le cr^{-2}\Big(r^{-2}\|D(\nabla_x)u;L_2(\Xi\cup\Upsilon_r)\|^2+\|D(\nabla_x)u;L_2(\Omega)\|^2\Big)\le cr^{-4}\|D(\nabla_x)u;L_2(\varpi)\|^2. \tag{5.15}$$

As the term $|w_3|^2$ in the integrand of Eq. (5.7) does not include the factor r^2, the majorant in the estimate of the component b_3 of the column b decreases:

$$|b_3|^2\le cr^{-4}\Big(\int\limits_{\Upsilon_r^0}(|w_3|+|u_3^\perp|)\,dx\Big)^2\le cr^{-2}\Big(\|D(\nabla_x)u;L_2(\Xi\cup\Upsilon_r)\|^2+\|D(\nabla_x)u;L_2(\Omega)\|^2\Big)\le cr^{-2}\|D(\nabla_x)u;L_2(\varpi)\|^2.$$

In addition, similar to Eq. (5.15), we obtain

$$|b_{6-p}|^2\le cr^{-4}\Big(\int\limits_{\Upsilon_r^0}\Big(|\tilde{x}_p|(|w_3|+|u_3^\perp|)+|x_3-R|(|w_p|+|u_p^\perp|)\Big)\,dx\Big)^2$$

$$\le cr^{-2}\Big(r^{-2}\|D(\nabla_x)u;L_2(\Xi\cup\Upsilon_r)\|^2+\|D(\nabla_x)u;L_2(\Omega)\|^2\Big)\le cr^{-4}\|D(\nabla_x)u;L_2(\varpi)\|^2.$$

Thereby, we have $\tilde{x}_1=x_1-1/2$ and $\tilde{x}_2=x_2$.

Finally, the component b_6 requires individual processing. It was found [37] that the Korn inequality (5.7) for the rod is insufficient for correct estimation of the torsion component of the strains, and the following additional inequality was established (see also Statement 3.4.13 in [2]):

$$\|w_p-\overline{w}_p;L_2(\Upsilon_r)\|^2\le c\|\partial_3(w_p-\overline{w}_p);L_2(\Upsilon_r)\|^2\le C\|D(\nabla_x)w;L_2(\Upsilon_r)\|^2. \tag{5.16}$$

Here, $p=1,2$ and $\overline{w}_p(x_3)$ is the mean of the function w_p over the rod cross section:

$$\overline{w}_p(x_3)=(\mathrm{mes}_2\omega_r)^{-1}\int\limits_{\omega_r}w_p(x)\,dx_1\,dx_2.$$

Applying estimates (5.5), (5.6), (5.16), and (5.11), we obtain

$$|b_6|^2\le cr^{-8}\Big(\Big(\int\limits_{\Upsilon_r^0}\{\tilde{x}_2[w_1(x)-\overline{w}_1(x_3)]-\tilde{x}_1[w_2(x)-\overline{w}_2(x_3)]\}\,dx\Big)^2+\Big(\int\limits_{\Upsilon_r^0}r(|u_1^\perp(x)|+|u_2^\perp(x)|)\,dx\Big)^2\Big)$$

$$\le cr^{-8}\mathrm{mes}_3(\Upsilon_r^0)\max\{|\tilde{x}_p|\colon x\in\overline{\Upsilon_r^0},\ p=1,2\}(\|D(\nabla_x)w;L_2(\Upsilon_r)\|^2+\|D(\nabla_x)u;L_2(\Omega)\|^2)$$

$$\le cr^{-4}\|D(\nabla_x)u;L_2(\varpi)\|^2. \tag{5.17}$$

Note that the subtrahends elements $\overline{w}_p(z)$ in Eq. (5.16) could be inserted into the first integrand of Eq. (5.17) by virtue of symmetry of the rod cross section Υ_r^0: the integral of the quantity $\tilde{x}_p$ over the ball ω_r equals zero.

Thus, the following inequality is obtained for the components of expansion (5.8) on the "cap" Ω:

$$r^4|b'|^2+r^2|b_3|^2+\|u^\perp;H^1(\Omega)\|^2\le c\|D(\nabla_x)u;L_2(\varpi)\|^2. \tag{5.18}$$

Here, $b'=(b_1,b_2,b_4,b_5,b_6)^{\mathrm{t}}$; correspondingly, the 3×5 matrix $d'(x)$ is obtained by crossing out the third column from matrix (5.9).

Thus, if the complex-valued vector function $u \in \overset{0}{H}{}^1(\varpi; \gamma^u)$ satisfies five orthogonality conditions

$$h' := \int\limits_{\Omega} d'\Big(x_1 - \frac{1}{2}, x_2, x_3 - R\Big)^{\mathrm{t}} u(x)\, dx = 0 \in \mathbb{C}^5, \tag{5.19}$$

then the column b' vanishes, and inequality (5.18) yields the relation

$$\begin{gathered} r^2 \|u; H^1(\Omega)\|^2 \le 2r^2 \|u^\perp; H^1(\Omega)\|^2 + 2r^2 |b_3|^2 \\ \le c\Big(r^2 \|D(\nabla_x)u; L_2(\Omega)\|^2 + \|D(\nabla_x)u; L_2(\varpi)\|^2\Big) \le c_\varpi \|D(\nabla_x)u; L_2(\varpi)\|^2. \end{gathered} \tag{5.20}$$

Indeed, according to Eqs. (5.8) and (5.10), we have

$$b = \Big\{ \int\limits_{\Omega} d\Big(x_1 - \frac{1}{2}, x_2, x_3 - R\Big)^{\mathrm{t}} d\Big(x_1 - \frac{1}{2}, x_2, x_3 - R\Big)\, dx \Big\}^{-1} h', \tag{5.21}$$

but by virtue of the definition of matrix (5.9) and the axial symmetry of the body Ω, the Gram matrix in braces in Eq. (5.21) [similar to the Gram matrix (5.13)] acquires a block-diagonal structure, i.e., in the sixth row and sixth column, it is only the diagonal element that differs from zero. Thus, equalities (5.19) and (5.21) mean that $b' = 0$.

Let us denote the subspace of the vector functions $u \in \overset{0}{H}{}^1(\varpi; \gamma^u)$ satisfying conditions (5.19) by $H_\perp$. Estimates (5.1), (5.7), and (5.20) yield the Korn inequality

$$r^2 \|u; H^1(\varpi)\|^2 \le c_\perp \|D(\nabla_x)u; L_2(\varpi)\|^2, \qquad u \in H_\perp, \tag{5.22}$$

where the factor $c_\perp$ is independent of the parameter $h \in (0, 1]$.

Note that the axial symmetry of the "cap" Ω used above is not obligatory: integration in Eq. (5.21) can be performed over a body of revolution inside Ω by adding the component $u^\perp$ already estimated in Eq. (5.11) to the integrand.

6. Identification of the Gap. Let us use the max-min principle (see, e.g., Theorem 10.2.2 in [30]) for the operator $-K(\eta)$:

$$-M_p(\eta) = \max_{H_p(\eta) \subset H(\eta)} \inf_{U \in H_p(\eta) \setminus \{0\}} \frac{\langle -K(\eta)U, U\rangle_\eta}{\langle U, U\rangle_\eta}. \tag{6.1}$$

Here, $p \in \mathbb{N}$ and $H_p(\eta)$ is an arbitrary subspace in $H(\eta)$ with the co-dimension $p - 1$, in particular, $H_1(\eta) = H(\eta)$. Let $p = 6$. As $H_6(\eta)$, we use a particular subspace

$$H_\perp(\eta) = \Big\{ U \in \overset{0}{H}{}^1_{\mathrm{per}}(\varpi; \gamma^u) \colon \quad u = \exp(i\eta x_1)U \in H_\perp \Big\},$$

decreasing thereby the right side of Eq. (6.1). Taking into account definitions (4.6) and (4.7) and inequality (5.22) for the vector function $u = \exp(i\eta x_3)U$, which yields the inequality

$$r^2 \|U; L_2(\varpi)\|^2 \le c_\perp \|D(\partial_1 + i\eta, \partial_2, \partial_3)U; L_2(\varpi)\|^2,$$

we obtain

$$M_6(\eta) \le -\inf_{U \in H_\perp(\eta)} \frac{-(\rho_0 U, U)_\varpi}{(A^0 D(\partial_1 + i\eta, \partial_2, \partial_3)U, D(\partial_1 + i\eta, \partial_2, \partial_3)U)_\varpi} \le \frac{C_{\rho_0}}{c_{A^0} c_\perp^{-1} r^2}. \tag{6.2}$$

Therefore, we found a constant $C_\varpi > 0$ for which the following relations are valid:

$$M_6(\eta) \le C_\varpi^{-1} r^{-2}, \qquad \Lambda_6(\eta) \ge C_\varpi r^2. \tag{6.3}$$

Let us prove that

$$M_5(\eta) \ge c_\varpi^{-1} r^{-4}, \qquad \Lambda_5(\eta) \le c_\varpi r^4. \tag{6.4}$$

For this purpose, we determine special displacement fields that vanish on the domain Ξ in the elastic union (5.4), which are, therefore, periodic:

$$\begin{gathered} u^p(x) = e_p \chi(x_3) - \tilde{x}_p e_3 \partial_3 \chi(x_3), \qquad u^{p+2}(x) = e_p x_3 \chi(x_3) - \tilde{x}_p e_3 \partial_3 (x_3 \chi(x_3)), \\ u^5(x) = (\tilde{x}_2 e_1 - \tilde{x}_1 e_2)\chi(x_3). \end{gathered} \tag{6.5}$$

Here, $p = 1$ and 2; e_j are the orths of the axes x_j; $\tilde{x}_1 = x_1 - 1/2$ and $\tilde{x}_2 = x_2$; $\chi \in C^\infty(\mathbb{R})$ is the cut-off function equal to unity at $x_3 \geq R$ and to zero at $x_3 \leq 0$. It should be noted that each vector function in Eqs. (6.5) on the domain Ξ is a nontrivial rigid displacement; hence, the following inequality is valid:

$$\|u^q; L_2(\varpi)\|^2 \geq c_0 > 0, \qquad q = 1, 2, \ldots, 5.$$

Direct calculations show that only the following strains are nontrivial among all strains $\varepsilon_{jk}(u^q)$:

$$\varepsilon_{33}(u^p; x) = -\tilde{x}_p \partial_3^2 \chi(x_3), \qquad \varepsilon_{33}(u^{p+2}; x) = -\tilde{x}_p \partial_3^2 (x_3 \chi(x_3)),$$

$$\varepsilon_{3k}(u^5; x) = \varepsilon_{k^3}(u^5; x) = -(-1)^k \tilde{x}_{3-k} \partial_3 \chi(x_3), \qquad p = 1, 2, \quad k = 1, 2.$$

The derivatives of the cut-off function χ differ from zero only on a set with a diameter $O(r)$; moreover, $|\tilde{x}_p| \leq r$ at $x \in \Upsilon_r$. Thus, we have

$$\|D(\nabla_x) u^q; L_2(\varpi)\|^2 \leq C_0 r^4, \qquad q = 1, 2, \ldots, 5.$$

All subspaces $H_5(\eta)$ with a co-dimension of four contain a linear combination

$$U(x) = \exp(-i\eta x_3) u(x) = \exp(-i\eta x_3) \sum_{q=1}^{5} \alpha_q u^q(x) \tag{6.6}$$

of five specified, linearly independent vector functions; we can assume that $|\alpha_1|^2 + |\alpha_2|^2 + \ldots + |\alpha_5|^2 = 1$. Increasing the value of the expression whose maximum value is calculated by Eq. (6.1), we choose an appropriate particular trial function (6.6) and find

$$-M_5(\eta) \leq -\frac{\langle K(\eta) U, U \rangle_\eta}{\langle U, U \rangle_\eta} = -\frac{(\rho_0 u, u)_\varpi}{(A^0 D(\nabla_x) u, D(\nabla_x) u)_\varpi} \leq -\frac{c_{\rho_0} c_0}{C_{A^0} C_0 r^4} =: c_\varpi^{-1} r^{-4}. \tag{6.7}$$

Thus, relations (6.4) are proved.

The positive constants C_ϖ and c_ϖ in Eqs. (6.3) and (6.4) are independent of the geometric parameter r whose boundary of variation $r_0 > 0$ can be fixed to satisfy the inequality $c_\varpi r_0^4 < C_\varpi r_0^2$. As a result, at $r \in (0, r_0]$, the continuous spectrum has at least one gap $l_{56} \supset (c_\varpi r^4, C_\varpi r^2)$ between the segments t_5 and t_6 (see Fig. 2). The analysis performed does not allow us to establish the presence or absence of gaps between the segments $t_1, t_2, \ldots, t_5$. It should be noted that the formation of the gap l_{56} is largely caused by a special shape of the cell ϖ, rather than by specific elastic properties of the material, because the right sides of inequalities (6.2) and (6.7) contain the constants c_{A^0}, C_{A^0} and c_{ρ_0}, C_{ρ_0}, for which estimates (1.5) and (1.7) are valid, respectively, but not the stiffness matrix A^0 and density ρ_0. The use of the condition of a clamped infinite surface Γ^u is justified, because without this condition it is not possible to separate five eigenvalues $\Lambda_1(\eta) \leqslant \Lambda_2(\eta) \leq \ldots \leq \Lambda_5(\eta)$ in sequence (4.9), which are infinitesimal quantities $O(r^4)$ of a higher order than the remaining terms of the sequence.

By increasing the density of one of the massive bodies in the periodic family on the interval $(0, \min\{\Lambda_1(\eta)\colon \eta \in [0, 2\pi)\})$, which is free from the continuous spectrum, we can form an arbitrary, defined in advance number of eigenvalues of problem (1.6) (cf. the procedure developed in [13]). The issue of existence of the trapping modes generated by the eigenvalues of problem (1.6) on the segments $t_1, t_2, \ldots, t_5$ of the continuous spectrum (to the left of the found gap l_{56}) remains still unclear. A waveguide with the gap containing the eigenvalue has not been found either.

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